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## SOLUTIONS OF EXERCISES.

### ACKNOWLEDGMENTS.

Joseph Bowden, Jr. 325 ; H. Y. Benedict 326 ; Geo. R. Dean 322 ; W. H. Echols 322 ; A. Hall 327 ; J. E. Hendricks 325 ; Artemas Martin 314, 320 ; F. Morley 321 ; J. F. McCulloch 323 ; J. C. Nagle 326 ; W. B. Richards 282, 316, 322 ; W. O. Whitescarver 320 ; Chas. Yardley 320 ; De Volson Wood 324.

### 314\*

A CYLINDER, diameter  $2b$ , intersects a sphere, diameter  $2a$ , the surface of the cylinder passing through the centre of the sphere. Required the part of the volume of the sphere contained by the cylinder.

[*Artemas Martin*].

#### SOLUTION.

Taking the origin at the centre of the sphere its rectangular equation is

$$x^2 + y^2 + z^2 = a^2, \quad (1)$$

and that of the cylindric hole is

$$x^2 + y^2 = 2bx. \quad (2)$$

Also,

$$V = \iiint dx \, dy \, dz. \quad (3)$$

Let  $x = r \cos \varphi$ , and  $y = r \sin \varphi$  ; whence (1) and (2) become

$$r^2 + z^2 = a^2, \quad (4)$$

$$r = 2b \cos \varphi, \quad (5)$$

and

$$V = \iiint r \, dr \, d\varphi \, dz. \quad (6)$$

The limits of  $z$  are  $-\sqrt{a^2 - r^2}$  and  $+\sqrt{a^2 - r^2}$  ; of  $r$ ,  $2b \cos \varphi$  and 0 ; of  $\varphi$ ,  $\frac{1}{2}\pi$  and 0.

$$\begin{aligned} V &= 2 \int \int r \sqrt{a^2 - r^2} \, dr \, d\varphi, = -\frac{2}{3} \int (a^2 - r^2)^{\frac{3}{2}} d\varphi \\ &= \frac{4}{3} \int_0^{\frac{1}{2}\pi} a^3 d\varphi - \frac{4}{3} \int_0^{\frac{1}{2}\pi} (a^2 - 4b^2 \cos^2 \varphi)^{\frac{3}{2}} d\varphi. \end{aligned} \quad (7)$$

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\* The above solution is for the case  $a > 2b$ . The case  $a < 2b$  remains to be treated. The case  $a = 2b$  is old.—ED.

Let  $\varphi = \frac{1}{2}\pi - \theta$ ; then  $d\varphi = -d\theta$ ,  $\cos \varphi = \sin \theta$ , and (7) becomes

$$\begin{aligned} V &= \frac{2}{3} \pi a^3 - \frac{4}{3} \int_0^{\frac{1}{2}\pi} (a^2 - 4b^2 \sin^2 \theta)^{\frac{3}{2}} d\theta \\ &= \frac{2}{3} \pi a^3 - \frac{4}{3} a^2 \int_0^{\frac{1}{2}\pi} \sqrt{a^2 - 4b^2 \sin^2 \theta} d\theta \\ &\quad + \frac{16}{3} b^2 \int_0^{\frac{1}{2}\pi} \sin^2 \theta \sqrt{a^2 - 4b^2 \sin^2 \theta} d\theta \\ &= \frac{2}{3} \pi a^3 + \frac{4}{3} a^3 \int_0^{\frac{1}{2}\pi} \sqrt{1 - e^2 \sin^2 \theta} d\theta \\ &\quad + \frac{16}{3} ab^2 \int_0^{\frac{1}{2}\pi} \sin^2 \theta \sqrt{1 - e^2 \sin^2 \theta} d\theta. \end{aligned}$$

Putting  $e = 2b/a$  we have

$$\begin{aligned} V &= \frac{2}{3} a^3 \pi - \frac{16}{9} a (a^2 - 2b^2) E \left[ \frac{1}{2} \pi, \frac{2b}{a} \right] \\ &\quad - \frac{4}{9} a (a^2 - 4b^2) F \left[ \frac{1}{2} \pi, \frac{2b}{a} \right]. \end{aligned}$$

When  $b = \frac{1}{2}a$ ,  $V = \frac{2}{3}a^3 \left[ \pi - \frac{4}{3} \right]$  [Artemas Martin].

321

A CIRCLE meets a hypocycloid of class 3 at six finite points. Show that the tangents to the hypocycloid at these six points touch a conic.

[Frank Morley.]

SOLUTION.

The equations of the hypocycloid in circular coordinates are

$$\left. \begin{aligned} x/c &= 2/t - t^2 \\ y/c &= 2t - 1/t^2 \end{aligned} \right\}, \quad (1)$$

$t$  being a complex quantity of modulus 1. (See a paper on the Epicycloid, American Journal, Vol. XIII, No. 2).

Any circle is

$$xy + \alpha x + \beta y + \gamma = 0.$$

Substituting from (1), we have a sextic to determine  $t$ , and we observe that the product of the roots is 1.

The tangent to (1) at  $t$  is

$$ux + vy + 1 = 0,$$

where

$$u = -t/c(1 + t^3), \quad v = -t^2/c(1 + t^3).$$

Let the line equation of a conic be

$$(A, B, C, F, G, H)(u, v, 1)^2 = 0.$$

Substituting for  $u, v$  in terms of  $t$ , we have again

$$\prod_1^6 t_r = 1,$$

$\prod$  denoting a product. This condition ensures that the six tangents touch a conic; and we saw that it holds if the points  $t$  are concyclic.

[*Frank Morley.*]

### 322

THE arc of a limaçon is shown in works on the Calculus to be equivalent to the arc of a certain ellipse. Show that the double point on the limaçon corresponds with Fagnani's point on the ellipse. [*W. B. Richards.*]

#### SOLUTION.

It is shown in works on the Calculus that

$$S = 2 \int_0^{\frac{1}{2}\pi} \{(a+b)^2 \cos^2 \varphi + (a-b)^2 \sin^2 \varphi\}^{\frac{1}{2}} d\varphi,$$

is the quadrant of the ellipse on semi-axes  $2(a+b)$  and  $2(a-b)$ , and also half of the whole length of the limaçon  $r = a \cos \theta + b$ .

It is well known that Fagnani's point divides the first arc into parts whose difference is  $4b$ , while the half difference between the two loops of the limaçon is also  $4b$ . [*W. H. Echols.*]

NOTE.—George R. Dean also points out that at Fagnani's point  $\cos \varphi = \sqrt{\frac{a+b}{2a}} = \cos \frac{1}{2}\theta$  at the node of the limaçon.—ED.

### 323

FOR solution see *Analyst*, Vol. I, No. 1, pp. 8–9. I proposed the problem in the *Schoolday Visitor Magazine* for May, 1872, nearly two years before Mr. Siverly used it in the *Analyst*. I had forgotten these facts when I sent it for publication in the ANNALS. [*Artemas Martin.*]